Encoding Higher-Order Logic in Spatio-Temporal Hypergraphs for Neuro-Symbolic Learning

Bikram Pratim Bhuyan[1], Amylia Ait Saadi[1,2] and Amar Ramdane-Cherif[1] *

1- LISV Laboratory, Université Paris-Saclay
10-12 Avenue of Europe, Vélizy, 78140, France

2- ESME Sudria, 38 rue Molière,
94200 Ivry sur seine, France

Abstract. This work integrates Monadic Second-Order (MSO) logic into Spatio-Temporal Heterogeneous Hypergraphs (STHH) to advance Neuro-Symbolic AI. By bridging higher-ordered symbolic logic with neural computations, STHH offers a novel framework for knowledge representation and learning. Evaluations on a custom agricultural dataset show that the proposed STHH outperforms state-of-the-art hypergraph models across F1-score, accuracy, and AUC metrics. Despite challenges such as limited standardized datasets, this study underscores the potential of integrating higher-ordered symbolic logic into neural systems to achieve robust and interpretable AI.

1 Introduction

Artificial intelligence (AI) aims to emulate human cognitive abilities, yet bridging the interpretability of symbolic logic with the adaptability of neural networks remains a core challenge. Neuro-Symbolic AI seeks to address this by enabling systems that are both interpretable and flexible [1, 2]. Despite progress, effectively embedding expressive logic within neural paradigms is still underexplored.

Monadic Second-Order (MSO) logic, with its ability to quantify over both individual elements and sets, offers a level of expressiveness beyond First-Order Logic (FOL) [3]. This makes it particularly suitable for capturing complex relationships such as graph connectivity, Hamiltonian paths, and k-colorability. Additionally, MSO's decidability on finite structures provides a computational framework for advanced reasoning [4]. However, embedding MSO logic into neural models presents challenges, especially in representing such logic efficiently while preserving its learning power.

Spatio-temporal heterogeneous hypergraphs (STHH) extend traditional knowledge graphs by incorporating vertices and hyperedges of varying types, as well as spatial and temporal properties. In this paper, we propose a structured representation that works with neural computation by mapping MSO logic into STHH, leveraging hypergraph convolutional networks tailored for directed structures that fill in gaps in interpretation and expression. Our contributions are three-fold: a) We encode MSO rules within the STHH framework, enabling complex

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reasoning in domains with spatio-temporal characteristics. b) We redefine spectral convolution for directed hypergraphs and integrate domain knowledge with data-driven learning via attention mechanisms. c) Using a custom dataset from the agricultural domain, we demonstrate the model's superiority over state-of-the-art methods in link prediction tasks.

2 Background and Related Works

Knowledge graphs (KGs) have emerged as a fundamental tool for structured knowledge representation, encoding relationships between entities in a graph structure. Current KG approaches can be broadly classified into symbolic logic-based methods and embedding-based techniques (KGEs).

Several methods have been proposed to incorporate logical rules (first-order) into the learning process of graph embeddings. For example, PTransE and its variants [5] integrate path rules into the embedding learning process to better capture relational paths in KGs. Similarly, methods like KALE and RUGE [6] use logical rules to regularize the embeddings. Inductive reasoning models, such as ILR-IR [7], incorporate manually defined logical rules like symmetry and transitivity into temporal knowledge graphs. Additionally, LogicENN and other methods [8] focusing on the joint learning of structure and rules have shown that integrating logical rules enhances the performance of KGEs by preserving graph structure and improving explainability. Furthermore, SimRE [9] leverages contrastive learning with soft logical rules to refine KGEs. However, most existing methods are constrained by the expressiveness of FOL. The necessity of Monadic Second-Order logic in this context lies in its expressive power, which is not possible with First-Order logic.

Hypergraphs, which generalize graphs by allowing hyperedges to connect multiple vertices, provide a flexible framework for modeling multi-entity relationships. Recent advancements in Hypergraph Neural Networks (HGNNs) like [10, 11, 12, 13, 14] and LHP [15] leverage spectral convolution and attention mechanisms to capture higher-order dependencies. However, these methods primarily focus on undirected hypergraphs; additionally, they do not integrate higher-order logical reasoning.

3 Encoding MSO Logic in STHH

A Spatio-Temporal Heterogeneous Hypergraph (STHH) is formally defined as:

$$G_{STHH} = (V, E, \tau_v, \tau_e, \Delta, \Omega, \xi, \rho),$$

where V is the set of vertices and $E \subseteq 2^V$ is the set of hyperedges. τ_v and τ_e represent vertex and hyperedge types, respectively. The mappings $\Delta : V \to \tau_v$ and $\Omega : E \to \tau_e$ assign types to vertices and hyperedges. The function $\xi : V \to \mathbb{R}^d$ assigns spatial coordinates to vertices, where d represents the spatial dimension. The temporal function $\rho : (V \cup E) \to T$ assigns temporal attributes, where T is a temporal domain. Vertices $V_i \subseteq V$ that share the same type c_j, spatial position $p_k \in \mathbb{R}^d$, and temporal property $t_l \in T$ are connected via a hyperedge. Formally, $\forall v \in V_i,\ \Delta(v) = c_j,\ \xi(v) = p_k,\ \rho(v) = t_l$.

An MSO formula Φ operates on vertices, hyperedges, and their sets. MSO consists of: a) Predicates- $A(x)$, where x represents a single vertex or hyperedge, and A is a unary predicate. b) Logical connectives- $\neg, \vee, \wedge, \rightarrow$, representing negation, disjunction, conjunction, and implication. c) Quantifiers- Existential ($\exists$) and universal ($\forall$) quantification over elements ($v \in V$) and sets ($V_i \subseteq V$). To represent the MSO in an STHH, a mapping can be established between the elements, sets, and relations of the MSO and the vertices, subsets of vertices, and hyperedges of STHH, respectively as shown in Figure 1.

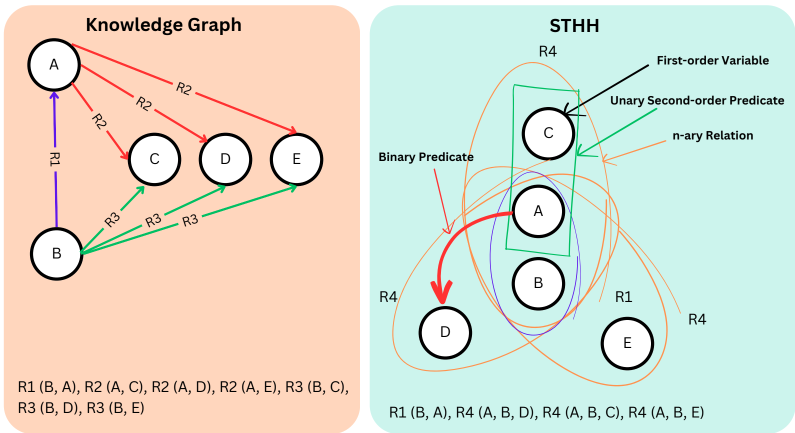

Fig. 1: Visual comparison of a traditional Knowledge Graph and its transformation into a Spatio-Temporal Heterogeneous Hypergraph (STHH) encoded with MSO logic.

Let $\mathcal{F} = V \cup E$ denote the universe of discourse. A unary predicate $A(x)$ in MSO maps an element $x \in \mathcal{F}$ to a vertex $v \in V$. For $x \in \mathcal{F}$ satisfying $A(x)$: $\forall x \in \mathcal{F},\ A(x) \implies \exists v \in V,\ \Delta(v) = A,\ \xi(v) = \xi(x),\ \rho(v) = \rho(x)$, where $\Delta(v)$ assigns the vertex type, $\xi(v)$ denotes its spatial coordinates and $\rho(v)$ its temporal attribute. The function $f : \mathcal{F} \rightarrow V$ maps elements in $\mathcal{F}$ to vertices in V, preserving their properties. For a unary second-order predicate $X(x)$, representing a set $X \subseteq \mathcal{F}$, the corresponding subset of vertices in V is $V_X \subseteq V$. For any $x \in X$: $\forall x \in X,\ X(x) \implies \exists v \in V_X,\ \Delta(v) = X,\ \xi(v) = \xi(X),\ \rho(v) = \rho(X)$. Hyperedges $e \in E$ connect subsets of vertices associated with X: $\forall X \subseteq \mathcal{F}, \exists e \in E, \forall x \in X, f(x) \in e,\ \Omega(e) = X$.

Logical connectives in MSO are assigned to vertex and edge operations in STHH, and second-order quantification extends existential (e.g. $\exists v\Phi$) and universal (eg. $\forall v\Phi$) quantification to subsets $V_X \subseteq V$, ensuring that their associated hyperedges satisfy Φ (MSO formula). An n-ary relation $R(x_1, x_2, \ldots, x_n)$ in MSO maps to a hyperedge $e \in E$ in STHH. For R: $\forall \{x_1, x_2, \ldots, x_n\} \subseteq \mathcal{F},\ R(x_1, x_2, \ldots, x_n) \implies \exists e \in E,\ \{f(x_1), f(x_2), \ldots, f(x_n)\} \subseteq e,\ \Omega(e) = R$.

Here, e captures the relationship R among the connected vertices.

The Hypergraph Convolutional Network (HCN) [16] operates directly on STHH, leveraging their non-Euclidean structure. Unlike traditional convolutional neural networks, HCNs handle complex relationships by utilizing the incidence matrix $\mathcal{H} \in \mathbb{R}^{|V| \times |E|}$ and corresponding degree matrices $(\mathcal{D}_V, \mathcal{D}_E)$. The convolution layer updates node embeddings as:
$\mathbf{X}^{(l+1)} = \sigma\left(\mathcal{D}_V^{-1/2}\mathcal{H}\mathcal{W}\mathcal{D}_E^{-1}\mathcal{H}^\top\mathcal{D}_V^{-1/2}\mathbf{X}^{(l)}\Theta^{(l)}\right)$, where $\mathbf{X}^{(l)}$ is the node embedding matrix at layer l, $\Theta^{(l)}$ is the learnable weight matrix, and σ is the activation function. To capture directional relationships, we adapt the Magnetic Laplacian framework [17] with Hermitian matrices for eigenvalue decomposition. The resulting spectral convolution integrates complex-valued edge features as $\mathbf{X} = \zeta f(\Lambda)\zeta^\dagger\mathbf{X}$, where ζ and Λ are eigenvectors and eigenvalues of the Hermitian matrix, respectively, and $f(\Lambda)$ is modeled using Chebyshev polynomials for computational efficiency.

Our methodology combines domain expert-driven and data-driven hypergraphs, denoted $STHH_X$ and $STHH_D$, respectively. Domain knowledge encodes MSO logic into $STHH_X$, while $STHH_D$ derives from spatio-temporal data. These hypergraphs are processed using HCNs with independent convolution layers as: $\mathbf{X}_X^{(l+1)} = \sigma\left(\mathcal{D}_{X,V}^{-1/2}\mathcal{H}_X\mathcal{W}_X\mathcal{D}_{X,E}^{-1}\mathcal{H}_X^\top\mathcal{D}_{X,V}^{-1/2}\mathbf{X}_X^{(l)}\Theta_X^{(l)}\right)$, and similarly for $\mathbf{X}_D$. Attention mechanisms combine the outputs as: $\Psi_V = \alpha_X\mathbf{X}_X + \alpha_D\mathbf{X}_D$, where α_X and α_D are learned attention weights. The embeddings Ψ_V are passed through dense layers and a softmax activation for link prediction. We employ a DistMult-based loss function for optimizing relational scores.

4 Experimental Evaluation

The proposed methodology was evaluated on a curated spatio-temporal dataset in an agricultural context representing complex multi-entity relationships. Our experiments aim to demonstrate the efficacy of encoding MSO logic into Spatio-Temporal Heterogeneous Hypergraphs (STHH) for link prediction tasks.

The domain expert-driven hypergraph ($STHH_X$) encodes 50 distinct MSO rules within the agricultural context. These rules encapsulate relationships like crop rotation, pest proximity, irrigation patterns, and spatio-temporal dependencies. For example, the rule $\exists A, B : \forall x \in A(crop(x, maize, year_X) \wedge \forall y \in B(crop(y, soybean, year_{X+1})))$ maps crop rotations across years into hyperedges where vertices represent fields. The first-order variable x is mapped to vertex V_i with type '*field*' and attribute '*maize*, $year_X$'. The set A is represented by a subset of vertices V with the type '*field*'. The hyperedge connects vertices in set A with vertices in set B representing '*soybean*, $year_{X+1}$'.

The data-driven hypergraph ($STHH_D$) was derived from the "FEW-Meter" project dataset [18]. The hypergraph statistics are shown in Table 1.

The dataset was split into 60% training, 20% validation, and 20% testing; the hidden layer size was set to 32, weight decay to $5e{-}4$, and the Adam optimizer with a learning rate of 0.0001 was employed. Table 2 provides a comparison

Table 1: Statistics of the $STHH_D$ hypergraph.

Statistic	**Value**
Number of Hyperedges	340
Number of Vertices	72
Average Degree of Vertices	15.69
Average Size of Hyperedges	3.32
Maximum Hyperedge Size	35
Minimum Hyperedge Size	1
Total Overlapping Hyperedge Pairs	3732

(ablation study) of our proposed STHH model with several state-of-the-art hypergraph neural network models with respect to the metrics: F1-score, Area Under the Curve (AUC), and Accuracy (ACC).

Table 2: Ablation Study: Impact of MSO Domain Rules Across Different Models

Model	**Setup**	**F1-Score**	**ACC**	**AUC**
HGNN [10]	Without Rules	0.501	0.629	0.665
	With Rules	0.532	0.662	0.693
HGNN+ [11]	Without Rules	0.672	0.689	0.707
	With Rules	0.698	0.713	0.722
H-GCN [12]	Without Rules	0.301	0.557	0.626
	With Rules	0.326	0.593	0.662
HNHN [13]	Without Rules	0.753	0.853	0.890
	With Rules	0.778	0.876	0.912
U-GNN [14]	Without Rules	0.763	0.816	0.887
	With Rules	0.791	0.844	0.913
LHP [15]	Without Rules	0.786	0.867	0.923
	With Rules	0.807	0.889	0.941
STHH (Proposed)	Without Rules	0.824	0.892	0.927
	With Rules	**0.856**	**0.922**	**0.951**

5 Conclusion

This work demonstrates the integration of Monadic Second-Order logic with Neuro-Symbolic systems through the construction of a Spatio-Temporal Heterogeneous Hypergraph. Our approach outperforms state-of-the-art models, achieving an F1-Score of 0.856, Accuracy of 0.922, and AUC of 0.951 in link prediction tasks. Despite these advancements, the lack of standardized datasets and toolkits for Neuro-Symbolic integration remains a challenge. Future efforts could focus on extensions beyond MSO logic; also, techniques such as low-rank approximations for spectral filtering, adaptive pruning of hyperedges, or quantization-based neural architectures could improve efficiency.

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